The calculation of the L and M, etc. critical absorption frequencies presents very great difficulties, for, if we suppose that an electron is removed from the second or third pair of orbits, it leaves this pair of orbits unbalanced. Just what would happen in this case is not clear, and it would require an additional assumption in order to complete the calculations. Definite general conditions of the dynamic equilibrium have not yet been found.

It may be, also, that orbits that are not circular would give better values than circular orbits. Computations of the frequencies on this basis present formidable difficulties. The fact, however, that the two quantum and three quantum orbits lie not in a plane, but in space of three dimensions may explain the appearance of three critical absorption wave-lengths in the L series, and six critical absorption wave-lengths in the M series, etc.

According to Sommerfeld's theory[3] the difference between two L absorption frequencies is due to the difference in shape of a circular and an elliptic orbit. His formula contains an undetermined constant. Professor Patterson and I have shown[4] that if we assume four electrons in the L orbit the undetermined constant is done away with, and that Sommerfeld's formula represents roughly the difference between the L_1 and L_2 absorption frequencies. It may be that a formula calculated on the basis of three dimensional orbits would give more accurate results.

I am greatly indebted to several of my assistants for carrying through many of the computations.

[1] These PROCEEDINGS, Sept., 1921, p. 260.

[2] *Nature*, March 24, 1921.

[3] *Atombau and Spektrallinien*, Chapter 5.

[4] These PROCEEDINGS, Sept., 1920, p. 517.

SEMI-COVARIANTS OF A GENERAL SYSTEM OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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It is known[1] that the most general transformation of the dependent variables which converts the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=o}^{m-1} \sum_{k=1}^{n} \binom{m}{l} p_{ikl}\, y_k^{(l)} = 0, \; (i = 1, 2, \ldots, n), \qquad (A)$$

where p_{ikl} are functions of the independent variable x, into another system of the same form is given by the equations

$$y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}\bar{y}_\lambda, \quad (k = 1, 2 \ldots, n), \tag{1}$$

where $\alpha_{k\lambda}$ are arbitrary functions of x and where the determinant Δ of the transformation does not vanish identically.

A function of the coefficients of (A) and their derivatives and of the dependent variables and their derivatives which has the same value for (A) as for any system derived from (A) by the transformation (1) is called a *semi-covariant*. If a semi-covariant does not contain the dependent variables or their derivatives, it is called a *seminvariant*. A complete system of seminvariants of system (A) has been calculated.[2] It is the purpose of this paper to obtain the additional semi-covariants necessary for a complete system of semi-covariants.

If equations (1) are solved for $\bar{y}_\lambda$, there results

$$\Delta\bar{y}_\lambda = \sum_{j=1}^{n} A_{j\lambda}\, y_j, \tag{2}$$

where $A_{j\lambda}$ is the algebraic minor of $\alpha_{j\lambda}$ in Δ. If the coefficients in (1) are assumed to satisfy the conditions for the transformation of (A) into the semi-canonical form,[2] the successive differentiation of (2) gives

$$\Delta\bar{y}_\lambda^{(\tau)} = \sum_{j=1}^{n} A_{j\lambda}\, t_{j\tau}, \quad (\tau = 1, 2, \ldots.), \tag{3}$$

where

$$t_{j\tau} = t'_{j,\,\tau-1} + \sum_{k=1}^{n} p_{j,\,k,\,m-1}\, t_{k,\,\tau-1}, \quad t_{jo} = y_j \tag{4}$$

The most general form of (1) which leaves the semi-canonical from in the semi-canonical form is given by the equations[2]

$$y_k = \sum_{\lambda=1}^{n} a_{k\lambda}\, \bar{y}_\lambda, \tag{5}$$

where $a_{k\lambda}$ are arbitrary constants whose determinant D is not zero. The semi-covariants in their semi-canonical form are obtained by transforming the semi-canonical form of (A) by (5). We shall let π_{ikl} denote the coefficients of the semi-canonical form of (A) which correspond to the

coefficients p_{ikl} of (A). The effect of the transformation (5) upon $\pi_{ikl}^{(\tau)}$ is given by the equations[2]

$$D\bar{\pi}_{ikl}^{(\tau)} = \sum_{\lambda=1}^{n}\sum_{\mu=1}^{n} A_{\lambda i}\, \pi_{\lambda\mu l}^{\tau}\, a_{\mu k}\ (l=0, 1, \ldots m-2) \tag{6}$$

where $A_{\lambda i}$ is the algebraic minor of $a_{\lambda i}$ in D.

If we put

$$r_i = \sum_{j=1}^{n} \pi_{i,j,m-2}\, r_{j,\,l-1}\ (i = 1, 2, \ldots n, l = 1, 2, \ldots n-1), \tag{7}$$

where $r_{jo} = y_j$, it is easily verified that each of the sets of quantities $r_{il} (i=1, 2, \ldots\ldots, n)$ is transformed by (5) cogrediently with y_i $(i=1, 2, \ldots\ldots, n)$. Therefore, the determinant

$$R = \begin{vmatrix} y_1, & y_2 \ldots\ldots\ldots\ldots & y_n \\ r_{11}, & r_{21} \ldots\ldots\ldots\ldots & r_{n1} \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ r_{1,n-1} & r_{2,n-1} \ldots\ldots & r_{n,n-1} \end{vmatrix}$$

is a relative semi-covariant.

Again, it is evident from (6) that

$$s_i = \sum_{j=1}^{n} \pi'_{i,j,m-2}\, y_j,\ (i=1, 2, \ldots n),$$

is a set of quantities which are transformed by (5) cogrediently with y_i $(i=1, 2, \ldots\ldots, n)$. We therefore have $n - 1$ additional relative semi-covariants

$$S_i = \sum_{j=1}^{n} s_j \frac{\partial R}{\partial r_{ji}},\ (i=1, 2, \ldots n-1).$$

Since the coefficients in (5) are constants, each set $y_i^{(\tau)}$ $(i=1,2,\ldots,n)$ of derivatives of y_i are transformed by (5) cogrediently with y_i $(i=1,2,\ldots,n)$. We therefore have $mn - n$ relative semi-covariants

$$T_{l\tau} = \sum_{j=1}^{n} y_j^{(\tau)} \frac{\partial R}{\partial r_{jl}},\ (l=0, 1, \ldots n-1; \tau=1, 2, \ldots, m-1)$$

A comparison of (3) with the inverse of (5) and of the expressions,[2] π_{ikl}, and π'_{ikl}, in terms of the coefficients of (A), with (6) shows that the

semi-covariants R, S_i, $T_{l\tau}$ may be expressed as semi-covariants of (A) simply by replacing $y_i^{(\tau)}$ by $t_{i\tau}$, π_{ikl} by u_{ikl}, and π_{ikl} by v_{ikl}, where u_{ikl} and v_{ikl} are functions of the coefficients of (A) and their derivatives which appear in the expressions for π_{ikl} and π'_{ikl}.

If the transformation (1) and the corresponding transformations for the derivatives of y_i are made infinitesimal, and the resulting system of partial differential equations for the semi-covariants is set up, it is found that there are exactly mn relative semi-covariants which are not seminvariants. We thus have the proper number of semi-covariants, but it remains to show that they are independent.

A comparison of R and S_i with the corresponding semi-covariants[3] for the special case of (A) where $m = 2$ shows R and S_i to be independent. Again, the functional determinant of $T_{l\tau}$ with respect to $y_i^{(\tau)}$ $(i=1,2,\ldots,n)$ for each value of $\tau = 1, 2, \ldots\ldots, m-1$ shows[3] that $T_{l\tau}$ are independent, among themselves and of R, of S_i and of the seminvariants.

We have now proved the following theorem:

All semi-covariants are functions of seminvariants and of R, S_i $(i = 1, 2,\ldots\ldots\ldots, n-1)$, $T_{l\tau}$ $(l = 0, 1,\ldots\ldots\ldots, n-1; \tau = 1, 2,\ldots\ldots\ldots m-1)$.

[1] Wilczynski, E. J., *Projective Differential Geometry of Curves and Ruled Surfaces*, Teubner, Leipzig, Chap. I.

[2] Stouffer, these PROCEEDINGS, **6**, 1920 (645–8).

[3] Stouffer, *London, Proc. Math. Soc.*, (Ser. 2), **17**, 1919 (337–52).

AN ALGORISM FOR DIFFERENTIAL INVARIANT THEORY

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1. Comprehensive as the existent theory of differential parameters is, as related to quantics

$$F = (a_0, a_1, \ldots, a_m)\ (dx_1, dx_2)^m\ (aj = aj(x_1, x_2)),$$

under arbitrary functional transformations

$$(1) \qquad x_i = x_i\,(y_1, y_2)\ (i = 1, 2),$$

developments of novelty relating to the foundations result when emphasis is placed upon the domains within which concomitants of such classes may be reducible, particularly a certain domain $R(\iota, T, \Delta)$ defined in part by certain irrational expressions in the derivatives of the arbitrary functions occurring in the transformations. For a given set of forms F all differential parameters previously known are functions in R of certain elementary invariants, which we designate as invariant elements, and their derivatives. The theory of invariant elements serves, therefore, to unify known theories and, for the various categories of parameters, gives a means of classification.